Symmedians*

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1 Introduction/Facts you should know

1. (Symmedian) Let ABC be a triangle and M be the midpoint of A. The reflection of AM over the angle bisector of A is the *symmedian* of A in triangle ABC.

2. (Symmedian point) The symmedians of a triangle are concurrent. This concurrency point is called the *symmedian point*.

3. (Isogonal conjugates) Let ABC be a triangle and P be a point not equal to any of A,B, C. The reflections of lines AP, BP, CP over the angle bisectors of A, B, C, respectively, concur at a point. This point is called the *isogonal conjugate of* ABC. The symmedian point is the isogonal conjugate of the centroid.

4. (Tangent construction) Let ABC be a triangle and let T be the intersections of the tangents at B and C to the circumcircle of ABC. Then AT is a symmedian.

5. (Symmedian and spiral similarity) Let ABC be a triangle with circumcenter O. Let X be the center of spiral similarity taking AB to CA. We have

   (a) AX is a symmedian,

   (b) $OX \perp AX$,

   (c) if AX is extended to meet the circumcircle of ABC at D, then X is the midpoint of AD,

   (d) If T is the intersection of the tangents to (ABC) at B and C, then $BTCX$ is cyclic.

6. (Symmedians in cyclic quads) Let $ABCD$ be a cyclic quadrilateral. The following are equivalent.

   (a) $AB \cdot CD = BC \cdot DA$.

   (b) AC is an A-symmedian of $\triangle DAB$.

*Some material and problems taken from http://yufeizhao.com/olympiad/three_geometry_lemmas.pdf

(c) AC is a C-symmedian of $\triangle BCD$.

(d) BD is a B-symmedian of $\triangle ABC$.

(e) BD is a D-symmedian of $\triangle CDA$.

2 Problems

1. (Poland 2000) Let ABC be a triangle with $AC = BC$, and P a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, then show that $\angle APM + \angle BPC = 180°$

2. (IMO Shortlist 2003) Given three fixed pairwise distinct points A, B, C lying on one straight line in this order. Let G be a circle passing through A and C whose center does not lie on the line AC. The tangents to G at A and C intersect each other at a point P. The segment PB meets the circle G at Q. Show that the point of intersection of the angle bisector of the angle AQC with the line AC does not depend on the choice of the circle G.

3. (Vietnam TST 2001) In the plane, two circles intersect at A and B, and a common tangent intersects the circles at P and Q. Let the tangents at P and Q to the circumcircle of triangle APQ intersect at S, and let H be the reflection of B across the line PQ. Prove that the points A, S, and H are collinear.

4. (APMO 2012) Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

5. (USAJMO 2011) Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $DE \parallel AC$. Prove that BE bisects AC.

6. (Russia 2009) In triangle ABC, let D be on side AC such that BD is the angle bisector of $\angle ABC$. The line BD intersects the circumcircle Ω of $\triangle ABC$ at B and E. Circle ω with diameter DE intersects Ω again at F. Prove that BF is a symmedian of $\triangle ABC$.

7. (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

8. Let $ABCD$ be a parallelogram which is not a rhombus. Let Q be the intersection point of the reflections of the lines AB and CD across the diagonals AC and DB respectively. Prove that Q is the center of the spiral similarity that maps segment AO to OD, where O is the center of the parallelogram.

9. Let $ABCD$ be a cyclic quadrilateral and M and N be the midpoints of the diagonals AC and BD. Prove that AC is the bisector of $\angle BMD$ if and only if BD is the bisector of $\angle ANC$.

10. (Balkan MO 2009). Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines BN and CM meet at point P. The circumcircles of triangles BMP and CNP meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.

11. (Balkan MO 2017) An acute angled triangle ABC is given, with $AB < AC$, and ω is its circumcircle. The tangents t_B, t_C at B,C respectively meet at L. The line through B parallel to AC meets t_C at D. The line through C parallel to AB meets t_B at E. The circumcircle of triangle BCD meets AC internally at T. The circumcircle of triangle BCE meets AB extended at S. Prove that ST, BC and AL are concurrent.

12. (MOP 1997) Let ABC be a triangle and D, E, F the points where its incircle touches sides BC, CA, AB, respectively. The parallel through E to AB intersects DF in Q, and the parallel through D to AB intersects EF in T. Prove that CF, DE, QT are concurrent.

13. (USA TST 2008) Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ||AB$ and $PR||AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

14. (USAMO 2008) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

15. Let A be one of the intersection points of circles ω_1, ω_2 with centers O_1, O_2. The line ℓ is tangent to ω_1, ω_2 at B, C respectively. Let O_3 be the circumcenter of triangle ABC. Let D be a point such that A is the midpoint of O_3D. Let M be the midpoint of O_1O_2. Prove that $\angle O_1DM = \angle O_2DA$.

16. (APMO 2013) Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.